Model Answer
B. Sc. 5th Semester Mathematics
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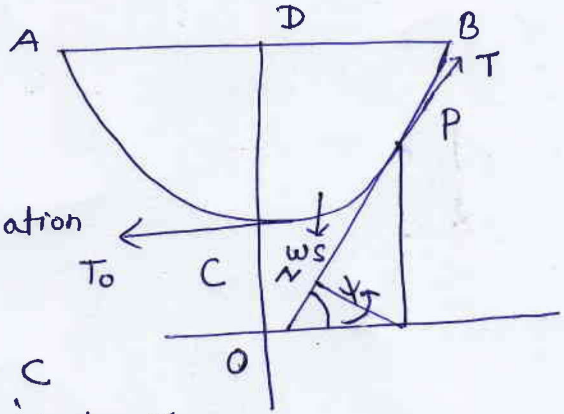
Subject: Mechanics-I **Paper Code: AS-2829**

1.(a). Let ACB be the string of which C is the lowest point. P is any point of the string such that the length of the arc CP = s. Let T be the tension at P and ψ its inclination with horizontal.



At the lowest point C tension is horizontal and let it be T_0.

Let w be weight per unit length. Then the portion CP of the string is in equilibrium under the action of the forces

(i) The tension T at P and T_0 at C.

(ii) The weight ws vertically downward

Resolving these forces horizontally and vertically, we have

$$T\cos\psi = T_0 \qquad \text{(I)}$$
$$T\sin\psi = ws \qquad \text{(II)}$$

Let the tension at lowest point T_0 be equal to the weight of the string of length c then

$$T_0 = wc \qquad \text{(III)}$$

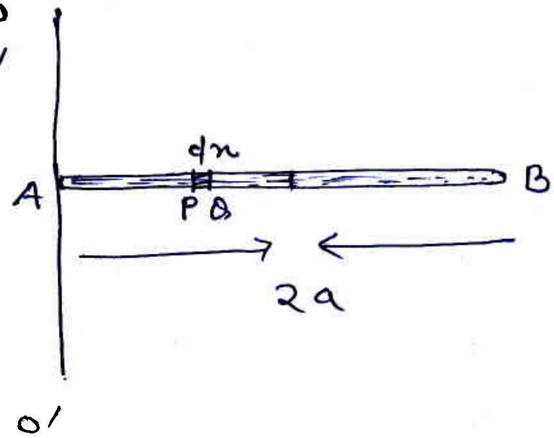
dividing (II) by (I) and putting value of T_0, we get

$$\tan\psi = s/c$$

$$\Rightarrow \boxed{s = c\tan\psi}$$

(b). Kinetically Equivalent system: Two mechanical systems that are such that their moments of inertia about all lines are the same, are called equimomental or Kinetically equivalent system.

Ans.



(c). Let AB be the rod and OAO' is the axis passing through one vertex A and perpendicular to the rod.

Let $AP = x$ and $PQ = dx$,

the mass of element $PQ = \dfrac{M}{2a}\,dx$.

Now moment of inertia of PQ about OAO'

$$= \left(\frac{M}{2a}\,dx\right) x^2$$

There-fore moment of inertia of whole rod about OAO'

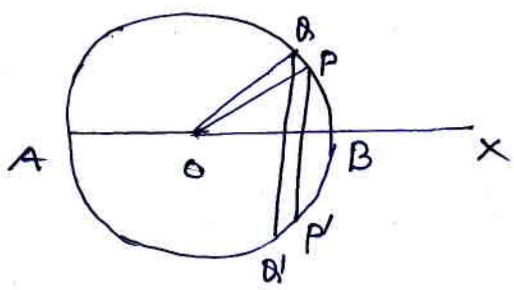
$$= \int_0^{2a} \frac{M}{2a} x^2\,dx$$

$$= \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a}$$

$$= \frac{M}{2a} \times \frac{8a^3}{3} = \frac{4Ma^2}{3}.$$

Ans.

(d). Let the surface of the sphere be divided into circular bands ~~lint~~ like $PQP'Q'$ perpendicular to Ox.



If $\angle POx = \theta$

and $\angle QOP = d\theta$

The surface of the band $= 2\pi a \sin\theta \cdot a\,d\theta$

Now moment of inertia about ox of the band

$$= \frac{M}{4\pi a^2}\left(2\pi a \sin\theta \cdot a\,d\theta\right) a^2 \sin^2\theta$$

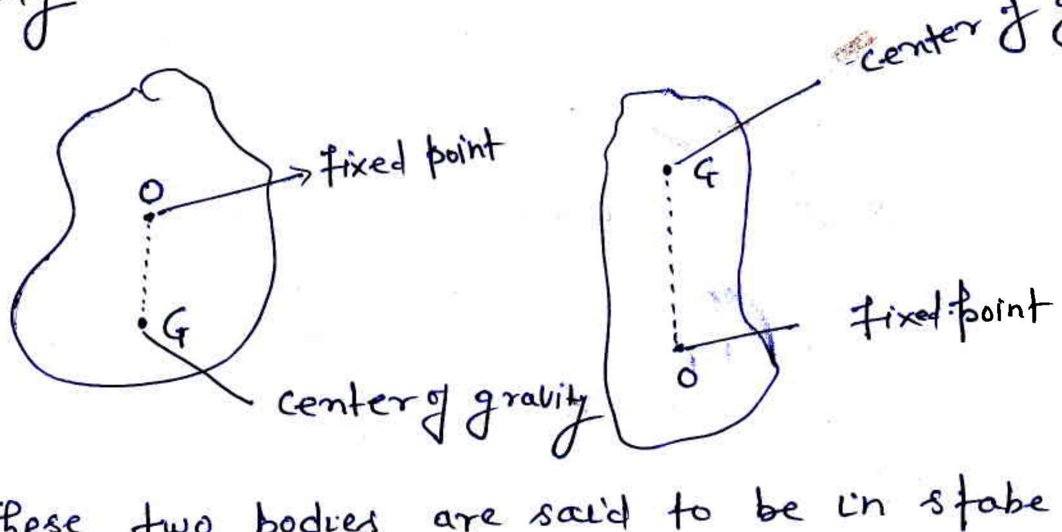
Hence, moment of inertia of the follow sphere about ox

$$= \frac{M' a^2}{2} \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= M \frac{2a^2}{3}.$$

Ans.

(e). Equilibrium of a rigid body! If a rigid body be in equilibrium, one point only of the body being fixed and the center of gravity of the body will be in the vertical line passing through the fixed point of the body.



These two bodies are said to be in stabe of and unstable equilibrium.

Energy test for Stability: We know that the sum of kinetic and potential energy is constant. In the position of equilibrium kinetic energy is zero, therefore potential energy is constant i.e potential energy is either maximum ~~of~~ or minimum. Now if the system be slightly displaced from a position of maximum potential energy and if the potential energy decreases i.e K.E. increases, or in other words kintetic energy will be positive as in the equilibrium position it was zero. Hence the system moves further away from the position of maximum potential energy showing that the equilibrium in the position of maximum potential energy is an unstable one. Fruther in the case of minimum potential energy ~~equ~~ we have stable equilibrium.

Maximum potential energy $\Rightarrow$ Unstable equilibrium

Minimum potential energy $\Rightarrow$ stable equilibrium.

For common catenary, We know that

$$y = c \sec \psi \quad \text{———} \quad (I)$$

and

$$s = c \tan \psi \quad \text{———} \quad (II)$$

From (I) and (II), We can Write

$$y^2 - s^2 = c^2 (\sec^2\psi - \tan^2 \psi)$$

$$= c^2$$

$$\Rightarrow \quad y^2 = c^2 + s^2$$

Ans.

Product of inertia: The product of inertia of a lamina with respect to the axes ox and oy is defined by $\int xy \, dm$, dm is elementry mass.

Since for any curve, we know that

$$\frac{dx}{ds} = \cos \psi \quad \text{———} \quad (I)$$

$$\frac{dy}{ds} = \sin \psi \quad \text{———} \quad (II)$$

Also we can write

$$\frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi}$$

$$\Rightarrow \quad \frac{dx}{d\psi} = \cos\psi \frac{d}{d\psi}(c\tan\psi), \quad \text{Since for catenary } s = c\tan\psi$$

$$= c \cos\psi \cdot \sec^2\psi$$

$$\Rightarrow \quad dx = c \sec\psi \, d\psi$$

Integrating both side, we have

$$x = c \log(\sec\psi + \tan\psi) + A \quad \text{——— (III)}$$

Since at $x = 0$, $\psi = 0$, From (III), we have

$A = 0$, putting in (III), we get

$$\boxed{x = c\log(\sec\psi + \tan\psi)}$$

Ans

2(b). We know that the equation of common catenary is

$$y = c\cosh\frac{x}{c}$$
$$= \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right)$$
$$= c\left[1 + \frac{1}{2!}\left(\frac{x}{c}\right)^2 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 + \cdots\right]. \quad \text{——— (I)}$$

Since span (x) is very small and c is very largy, therefore $\frac{x}{c}$ will be very small and the powers of $\frac{x}{c}$ beyond two can be neglected, so that equation (I) can be written as

$$y = c\left(1 + \frac{x^2}{2c^2}\right)$$

$$\Rightarrow \quad x^2 = 2c(y - c) \quad \text{——— (II)}$$

Equation (II) represents a parabola with latus rectum $2c$ or $\frac{2T_0}{w}$, where w is weight per unit length and $T_0 (= wc)$ is tension at the lowest point.

Ans.

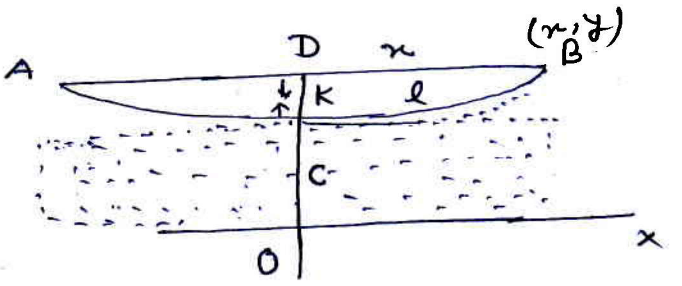
(3.).

ⓐ ⓑ

ACB is the chain of length l.

AB is breath of the river,

and CD = k (given)



We know that

$$y = c\cosh x/c \quad \text{—————} \quad (i)$$

At the point B, $y = c+k$, therefore we have

$$c+k = c\left[1 + \frac{x^2}{2c^2} + \frac{1}{3!}\frac{x^3}{c^3} + \cdots\right]$$

$$c+k = c + \frac{x^2}{2c} \qquad \left(\tfrac{x}{c} \text{ is very small neglecting higher } x/c\right)$$

$$\Rightarrow \quad k = \frac{x^2}{2c}$$

$$\Rightarrow \quad c = \frac{x^2}{2k} \quad \text{—————} \quad (ii)$$

Now we know that

$$s = c\sinh x/c$$

$$\frac{l}{2} = c\left[\frac{x}{c} + \frac{x^3}{3!\,c^3} + \cdots\right]$$

$$\Rightarrow \cancel{2}\frac{l}{\cancel{2}} = 2x + \frac{x^3}{\cancel{6}_{3}\,c^2}\times\cancel{2}$$

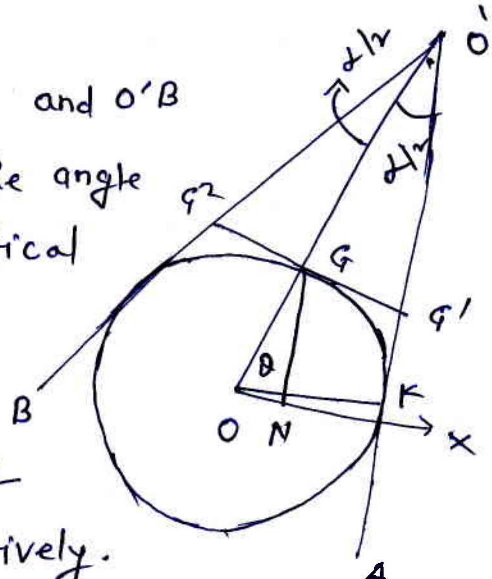
$$\Rightarrow \quad l - 2x = \frac{x^3}{3c^2} = \frac{x^3}{3}\,\frac{4k^2}{x^4} \qquad [\text{from (ii)}]$$

$$\Rightarrow \quad l - 2x = \frac{4k^2}{3x} = \frac{8}{3}\cdot\frac{k^2}{l} \qquad \text{Since } x \simeq l$$

⇒ Length of the chain − Breath of the river

$$= \frac{8}{3}\frac{k^2}{l}.$$

Ans.



(4). Two equal uniform rods O'A and O'B are joined at O' so that the angle between them is α in vertical plane on a smooth sphere of radius r. G^1 and G^2 are center of gravity of the rods O'A and O'B respectively.

Let G be their combined C.G. then O'G will bisect the angle BO'A and also the base G^1G^2.

Let $O'A = O'B = 2a$

Now the height of C.G. of the system above the horizontal

$$z = GN = (OG \sin\theta).$$

$$= (OO' - O'G)\sin\theta \qquad \text{——— (1)}$$

Now from right angle triangle O'OK and O'GG' we have

$$\sin\alpha/_2 = \frac{r}{OO'}, \qquad \cos\frac{\alpha}{2} = \frac{O'G}{O'G_1}$$

$$\Rightarrow \quad OO' = r\,\mathrm{cosec}\frac{\alpha}{2}, \quad O'G = a\cos\frac{\alpha}{2}$$

Putting these values in (1), we get

$$z = \left(r\,\mathrm{cosec}\frac{\alpha}{2} - a\cos\frac{\alpha}{2}\right)\sin\theta$$

$$\Rightarrow \quad \frac{dz}{d\theta} = \left(r\,\mathrm{cosec}\frac{\alpha}{2} - a\cos\frac{\alpha}{2}\right)\cos\theta$$

and $$\frac{d^2z}{d\theta^2} = \left(a\cos\frac{\alpha}{2} - r\operatorname{cosec}\frac{\alpha}{2}\right)\sin\theta$$

For position of equilibrium,

$$\frac{dz}{d\theta} = 0 \Rightarrow \left(a\cos\frac{\alpha}{2} - r\operatorname{cosec}\frac{\alpha}{2}\right)\cos\theta = 0$$

$$\Rightarrow \cos\theta = 0$$

$$\Rightarrow \theta = \pi/2$$

Now $$\left(\frac{d^2z}{d\theta^2}\right)_{\theta=\frac{\pi}{2}} = \left(a\cos\frac{\alpha}{2} - r\operatorname{cosec}\frac{\alpha}{2}\right)$$

Hence the equilibrium will be stable or unstable according as z is Min. or Max. i.e.

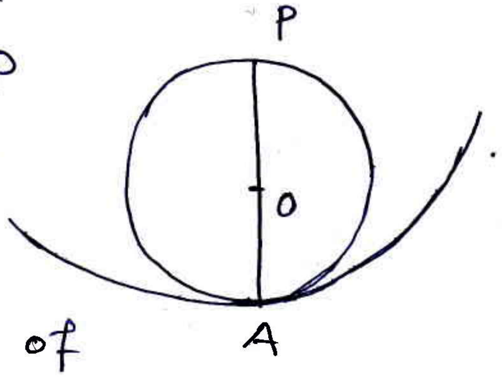
$$a\cos\frac{\alpha}{2} > \text{ or } < r\operatorname{cosec}\frac{\alpha}{2}$$

or $$2a\cos\frac{\alpha}{2} < \text{ or } < 2r\operatorname{cosec}\frac{\alpha}{2}$$

~~or 2a <~~

$$\Rightarrow 2a > \text{ or } < 4r\operatorname{cosec}\alpha .$$ Ans

(5). Let W be the weight of the sphere of radius r and let P the weight attached to its ~~lowest~~ highest point. If h be the height of the C.G. of the system above A, the point of contact of the sphere and the bowl,



$$h = \frac{W.r + P.2r}{W+P}$$

For stable equilibrium

$$\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$$

where R is the radius of the bowl (here r and R are measured in opposite directions).

$$\therefore \quad \frac{W+P}{r(W+P)} > \frac{1}{r} - \frac{1}{2r} \quad , \quad \text{since } R = r$$

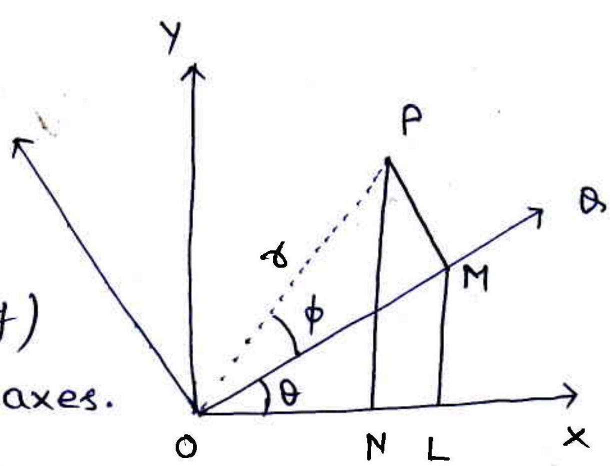
or $\quad W + P > \frac{1}{2}(W+2P)$

or $\quad W > \frac{W}{2}$

Which is always true and independent of P. Hence the equilibrium is stable for all values of P.

(6).

Let m be the element of mass of the lamina enclosing a point P(x, y) referred to ox, oy as axes.



Let OQ is any line making an angle θ with ox.

Let A, B be the moments of inertia of the lamina about ox and oy and F, its product of inertia about ox, oy. Then, if PM is the perpendicular from P on OQ, moment of inertia of the lamina about

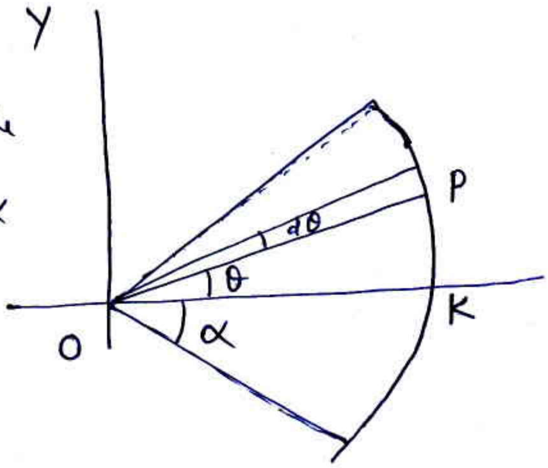
$$
\begin{aligned}
OQ &= \Sigma m\,(PM)^2 \\
&= \Sigma m\,(y\cos\theta - x\sin\theta)^2 \\
&= \Sigma \cos^2\theta\,(my^2) + \Sigma\sin^2\theta\,(mx^2) \\
&\qquad - 2\sin\theta\cos\theta\,\Sigma mxy \\
&= \cos^2\theta\,\Sigma my^2 + \sin^2\theta\,\Sigma mx^2 \\
&\qquad - 2\sin\theta\cos\theta\,\Sigma mxy \\
&= A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta
\end{aligned}
$$

Now the product of inertia about OQ and OR

$$
\begin{aligned}
&= \Sigma m\; OM\cdot PM \\
&= \Sigma m\,(x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta) \\
&= \sin\theta\cos\theta\,\Sigma my^2 - \sin\theta\cos\theta\,\Sigma mx^2 \\
&\qquad + (\cos^2\theta - \sin^2\theta)\,\Sigma mxy \\
&= (A-B)\sin\theta\cos\theta + F\cos 2\theta
\end{aligned}
$$

Ans

7. Let O be the centre and K the middle point of the arc subtending an angle 2α at O. If M be the mass of ~~an element~~ the arc of the circle of radius a.



Mass per unit length $= \dfrac{M}{2a\alpha}$

Mass of an element $a\,d\theta$ at $P = \dfrac{M}{2a\alpha}\, a\,d\theta$

Hence moment of inertia about OK $= \displaystyle\int_{-a}^{a} \frac{M}{2a\alpha}\, a\,d\theta\; a^2 \sin^2\theta$

and moment of inertia about oy $= \displaystyle\int_{-a}^{a} \frac{M}{2a\alpha}\, a\,d\theta\; a^2 \cos^2\theta$

therefore moment of inertia about an axis through O perpendicular to the plane = sum of moments of inertia about OK and about oy

$$= \int_{-a}^{a} \frac{M}{2a\alpha}\left(a^2\sin^2\theta + a^2\cos^2\theta\right) a\,d\theta$$

$$= M a^2$$

Ans

Q- (a). Compound Pendulum: A compound pendulum is one where the rod is not massless, and may have extended size, that is, an arbitrarily shaped rigid body swinging by a pivot. In this case pedulum pediods depends on its moment of inertia I around the pivot point.

The equation of torque is given by

$$\tau = I\alpha,$$

where α is the angular acceleration, τ is the torque. ~~[illegible]~~

Since torque is generated by gravity so,

$$\tau = -mg L \sin\theta$$

Where L is the distance from the pivot to the centre of mass of pedulum. θ is the angle from the verticle.

For small angle approximation $\sin\theta \simeq \theta$

$$\alpha \simeq \frac{mg L\theta}{I}$$

This gives a period of

$$\boxed{T = 2\pi \sqrt{I/mgL}}$$

Ans

(b). D'alembert Principle: The principle states that the sum of the differences between the forces acting on a system of mass particles and time derivatives of the momenta of the system itself along any virtual displacement consistent with the constraints of the system, is zero. Symbolically we can written as

$$\sum_i (F_i - m_i a_i) \cdot \delta r_i = 0,$$

Where

i is an integer

F_i is the total applied force on ith particle

m_i is mass of ith particle

a_i is acceleration of ith particle

δr_i is vertual displacement of the ith particle

This principle is also known as Lagrange's D'Alembert principle. It is named after its discoverer, the French Physicist and Mathematician Jean le Rond d'Alembert.

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